

NONAUTONOMOUS EQUATIONS, GENERALIZED DICHOTOMIES AND STABLE MANIFOLDS

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ABSTRACT. Assuming the existence of a general nonuniform dichotomy for the evolution operator of a non-autonomous ordinary linear differential equation in a Banach space, we establish the existence of invariant stable manifolds for the semiflow generated by sufficiently small nonlinear perturbations of the linear equation. The family of dichotomies considered satisfies a general growth rate given by some increasing differentiable function, allows situations for which the classical Lyapunov exponents are zero, and contains the nonuniform exponential dichotomies as a very particular case. In addition we also give explicit examples of linear equations that admit all the possible considered dichotomies.

1. INTRODUCTION

The existence of invariant manifolds is an important subject in the theory of dynamical systems and differential equations. The fundamental tools to establish the existence of invariant manifolds are, for dynamical systems, the notion of nonuniform hyperbolicity and, for differential equations, the related notion of nonuniform dichotomy.

The concept of nonuniform hyperbolicity, introduced by Pesin [20, 21, 22], is a generalization of the classical concept of (uniform) hyperbolicity where the rates of expansion and contraction are allowed to vary from point to point. For nonuniformly hyperbolic trajectories, Pesin [20] was able to obtain a stable manifold theorem in the finite-dimensional setting. Since then, there were several contributions to the theory. Namely, in [24] Ruelle gave a proof of the stable manifold theorem based on the study of perturbations of products of matrices occurring in Oseledec's multiplicative ergodic theorem in [18]. Another proof, using graph transform techniques and based on the classical work of Hadamard, was given by Pugh and Shub in [23]. Following his approach in [24], Ruelle in [25] proved a stable manifold theorem in Hilbert spaces under some compactness assumptions. In [17] Mañé obtained a corresponding version for transformations in Banach spaces under some compactness and invertibility assumptions, that includes the case of differentiable maps with compact derivative at each point, and, in [26], Thieullen generalized the results of Mañé for a family of transformations satisfying some asymptotic compactness.

On the other hand, in the setting of nonautonomous differential equations, Barreira and Valls [5, 3] introduced the notion of nonuniform exponential dichotomy based on the classical notion of exponential dichotomy introduced by Perron in [19] and also in the notion of nonuniformly hyperbolic trajectory introduced by Pesin in [20, 21, 22]. Versions of the stable manifold theorems for nonuniformly exponential dichotomies were also obtained, both in the continuous and the discrete

time settings. In fact, for flows and semiflows arising from nonautonomous ordinary differential equations, Barreira and Valls were able to obtain stable manifold theorems in several contexts. For more details about the stability theory of nonautonomous differential equations with nonuniform exponential dichotomies and, in particular, the existence of invariant manifolds, the reader can consult the book [8]. Corresponding results were also obtained in the discrete time setting, namely in [4], Barreira and Valls obtained C^1 stable manifolds for nonuniformly exponential dichotomies in finite dimension and, using this result as a starting point, in [2] it was established the existence of C^k local manifolds for C^k perturbations by an induction process that uses a linear extension of the dynamics. For Banach spaces, assuming a nonuniform exponential dichotomy, it was established in [1] the existence of C^1 global stable manifolds for some perturbations of linear dynamics.

The purpose of this paper is to obtain global stable manifolds for perturbations of linear ordinary differential equations, assuming some general type of dichotomy for the evolution operator associated with the linear equation. This dichotomies bound the norms of the evolution operator by a nonuniform law that is not necessarily exponential. In fact, the dichotomies considered (see (2) and (3)) contain the nonuniform exponential dichotomies as a very particular case. The existence of global stable manifolds for perturbations of linear ordinary differential equations with nonuniform dichotomies that are not exponential was already addressed in [10] for the particular case of polynomial dichotomies (more precisely, in that paper the dichotomies considered correspond to the dichotomies obtained in the present case by setting $\mu(t) = t + 1$ in (2) and (3)). In the discrete time setting, this problem was discussed in [11] for perturbations of some nonuniform polynomial dichotomies.

The perturbations considered here include as a particular case the ones considered in [6] and [10]. Therefore, the theorems proved there, respectively for nonuniform exponential dichotomies and for nonuniform polynomial dichotomies, are particular cases of the result obtained in this paper. We emphasize that, contrarily to what happens in the exponential case, we need to prove inequality (41) without using Gronwall's Lemma. This was done in Lemma 2 by mathematical induction.

Notice that the family of dichotomies considered in this paper includes cases where the Lyapunov exponent considered in [8] for Hilbert spaces is zero for all $v \in E_1$ (see Section 2 for the definition of E_1). In fact, we only have finite nonzero Lyapunov exponent if the growth rate is, in some sense, close to exponential.

On the other hand, it is strait-forward to see that for a linear equation $u' = A(t)u$ and initial conditions u_0 the map $\chi : X \rightarrow [-\infty, +\infty]$ given by

$$\chi(u_0) = \limsup_{n \rightarrow +\infty} \frac{\log \|u(t)\|}{\log \mu(t)}$$

is a Lyapunov exponent in the sense of the abstract theory developed in [12]. In [7] Barreira and Valls used these Lyapunov exponents to establish the existence of generalized trichotomies for linear equations in finite dimension, assuming that the matrices $A(t)$ are in block form, and also that

$$\lim_{t \rightarrow +\infty} \frac{\log t}{\log \mu(t)} = 0,$$

where the increasing functions μ are the growth rates considered there.

In another direction, in a recent work, Barreira and Valls [9] established the persistence of generalized dichotomies under small linear perturbations. The dichotomies they consider correspond to the dichotomies considered here though the definition is given in slightly different manner, namely our definition correspond to the one given there setting $\mu(t) = e^{\rho(t)}$ (compare (2) and (3) with Section 3 of [9]).

The content of the paper is as follows: in section 2 we establish the setting and define the dichotomies to be considered; then, in section 3, we present, for each growth rate μ in our family of growth rates, examples of nonuniform μ -dichotomies that are not uniform μ -dichotomies; finally in section 4 we prove our result on the existence of stable manifolds for a large family of sufficiently small perturbations of the linear differential equations considered.

2. NOTATION AND PRELIMINARIES

Let $B(X)$ be the space of bounded linear operators acting on a Banach space X . We are going to consider the initial value problem

$$v' = A(t)v, \quad v(s) = v_s \quad (1)$$

with $s \geq 0$ and $v_s \in X$ and where $A: \mathbb{R}_0^+ \rightarrow B(X)$ is a C^1 function. We assume that each solution of (1) is global and we denote by $T(t, s)$ the evolution operator associated with (1), i.e., $v(t) = T(t, s)v_s$ for $t \geq 0$.

Let $\mu: \mathbb{R}_0^+ \rightarrow [1, +\infty[$ be an increasing differentiable function such that

$$\lim_{t \rightarrow +\infty} \mu(t) = +\infty.$$

We say that equation (1) admits a *nonuniform μ -dichotomy* in \mathbb{R}_0^+ if, for each $t \geq 0$, there are projections $P(t)$ such that

$$P(t)T(t, s) = T(t, s)P(s), \quad t, s \geq 0$$

and constants $D \geq 1$, $a < 0 \leq b$ and $\varepsilon \geq 0$ such that, for every $t \geq s \geq 0$,

$$\|T(t, s)P(s)\| \leq D \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^\varepsilon, \quad (2)$$

$$\|T(t, s)^{-1}Q(s)\| \leq D \left[\frac{\mu(t)}{\mu(s)} \right]^{-b} \mu(t)^\varepsilon, \quad (3)$$

where $Q(t) = \text{Id} - P(t)$ is the complementary projection. When $\varepsilon = 0$ we say that we have a *uniform μ -dichotomy* or simply a *μ -dichotomy*. We define for each $t \geq 0$ the linear subspaces

$$E(t) = P(t)X \quad \text{and} \quad F(t) = Q(t)X.$$

Without loss of generality, we always identify the spaces $E(t) \times F(t)$ and $E(t) \oplus F(t)$ as the same space and in these spaces we use the norm given by

$$\|(x, y)\| = \|x\| + \|y\|, \quad (x, y) \in E(t) \times F(t).$$

Hence, the unique solution of (1) can be written in the form

$$v(t) = (U(t, s)\xi, V(t, s)\eta), \quad t \geq s$$

where $v_s = (\xi, \eta) \in E(s) \times F(s)$ and

$$U(t, s) := P(t)T(t, s)P(s) \quad \text{and} \quad V(t, s) := Q(t)T(t, s)Q(s).$$

3. EXAMPLES

In the following family of examples we are going to present nonautonomous linear equations that admits a nonuniform μ -dichotomy for each possible function μ .

Example 1. *Given $\varepsilon > 0$ and $a < 0 \leq b$, consider the differential equation in \mathbb{R}^2 given by*

$$\begin{aligned} u' &= \left(a \frac{\mu'(t)}{\mu(t)} + \omega \frac{\mu'(t)}{\mu(t)} (\cos t - 1) - \omega \log \mu(t) \sin t \right) u \\ v' &= \left(b \frac{\mu(t)}{\mu'(t)} - \omega \frac{\mu'(t)}{\mu(t)} (\cos t - 1) + \omega \log \mu(t) \sin t \right) v \end{aligned} \quad (4)$$

where $\omega = \varepsilon/2$. The evolution operator associated with this equation is given by

$$T(t, s)(u, v) = (U(t, s)u, V(t, s)v),$$

where

$$\begin{aligned} U(t, s) &= \left[\frac{\mu(t)}{\mu(s)} \right]^a e^{\omega \log \mu(t)(\cos t - 1) - \omega \log \mu(s)(\cos s - 1)}, \\ V(t, s) &= \left[\frac{\mu(t)}{\mu(s)} \right]^b e^{-\omega \log \mu(t)(\cos t - 1) + \omega \log \mu(s)(\cos s - 1)}. \end{aligned}$$

Let $P(t): \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the projections defined by $P(t)(u, v) = (u, 0)$ and $Q(t) = \text{Id} - P(t)$. Then we have

$$\begin{aligned} \|T(t, s)P(s)\| &= |U(t, s)| \leq \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^\varepsilon \\ \|T(t, s)^{-1}Q(t)\| &= |V(t, s)^{-1}| \leq \left[\frac{\mu(t)}{\mu(s)} \right]^{-b} \mu(t)^\varepsilon \end{aligned}$$

and this shows that equation (4) admits a nonuniform μ -dichotomy.

Furthermore, since

$$U(2k\pi, 2k\pi - \pi) = \left[\frac{\mu(2k\pi)}{\mu(2k\pi - \pi)} \right]^a (\mu(2k\pi - \pi))^\varepsilon, \quad k \in \mathbb{N},$$

the nonuniform part can not be removed.

The examples in [10] are the particular case of this family of examples obtained by setting $\mu(t) = t + 1$.

4. STABLE MANIFOLDS

The purpose of this paper is the obtention of stable manifolds for the nonlinear problem

$$v' = A(t)v + f(t, v), \quad v(s) = v_s \quad (5)$$

when equation (1) admits a nonuniform μ -dichotomy and $f: \mathbb{R}_0^+ \times X \rightarrow X$ is a perturbation of class C^1 and there exists $\delta > 0$ such that, for every $t \geq 0$ and $u, v \in X$,

$$f(t, 0) = 0, \quad \partial f(t, 0) = 0, \quad (6)$$

$$\|\partial f(t, u)\| \leq \delta \mu'(t) \mu(t)^{-3\varepsilon-1}, \quad (7)$$

$$\|\partial f(t, u) - \partial f(t, v)\| \leq \delta \mu'(t) \mu(t)^{-3\varepsilon-1} \|u - v\|, \quad (8)$$

where, for a question of simplicity, ∂ denotes the partial derivative with respect to the second variable and $\mu(t)$ and ε are the same as in (2) and (3). A trivial application of the mean value theorem combined with (7) yields

$$\|f(t, u) - f(t, v)\| \leq \delta \mu'(t) \mu(t)^{-3\varepsilon-1} \|u - v\| \quad (9)$$

for every $u, v \in X$ and, with $v = 0$, equation (9) becomes

$$\|f(t, u)\| \leq \delta \mu'(t) \mu(t)^{-3\varepsilon-1} \|u\|. \quad (10)$$

For

$$G = \bigcup_{t \geq 0} \{t\} \times E(t) \quad (11)$$

we define the space \mathcal{X} of C^1 functions $\phi: G \rightarrow X$ such that

$$\phi(s, \xi) \in F(s), \quad (12)$$

$$\phi(s, 0) = 0, \quad \partial\phi(s, 0) = 0, \quad (13)$$

$$\|\partial\phi(s, \xi)\| \leq 1, \quad (14)$$

$$\|\partial\phi(s, \xi) - \partial\phi(s, \bar{\xi})\| \leq \|\xi - \bar{\xi}\|, \quad (15)$$

for every $(s, \xi), (s, \bar{\xi}) \in G$. By the mean value theorem and (14) we have

$$\|\phi(s, \xi) - \phi(s, \bar{\xi})\| \leq \|\xi - \bar{\xi}\| \quad (16)$$

for every $(s, \xi), (s, \bar{\xi}) \in G$ and putting $\bar{\xi} = 0$ in (16) we get

$$\|\phi(s, \xi)\| \leq \|\xi\|$$

for every $(s, \xi) \in G$.

For each $\phi \in \mathcal{X}$ we define the graph

$$\mathcal{V}_\phi = \{(s, \xi, \phi(s, \xi)) : (s, \xi) \in G\}. \quad (17)$$

Writing the unique solution of (5) in the form

$$(x(t, s, v_s), y(t, s, v_s)) \in E(t) \times F(t),$$

where $v_s = (\xi, \eta) \in E(s) \times F(s)$, we define for each $\tau \geq 0$ the semiflow given by

$$\Psi_\tau(s, v_s) = (s + \tau, x(s + \tau, s, v_s), y(s + \tau, s, v_s)). \quad (18)$$

Now we will formulate our theorem on the existence of global stable manifolds.

Theorem 1. *Let X be a Banach space, assume that equation (1) admits a nonuniform μ -dichotomy in \mathbb{R}_0^+ for some $D \geq 1$, $a < 0 \leq b$ and $\varepsilon > 0$, and let $f: \mathbb{R}_0^+ \times X \rightarrow X$ be a function satisfying (6), (7) and (8) for some $\delta > 0$. If*

$$a + \varepsilon < b, \quad (19)$$

then, choosing $\delta > 0$ sufficiently small, there exists a unique function $\phi \in \mathcal{X}$ such that

$$\Psi_\tau(\mathcal{V}_\phi) \subseteq \mathcal{V}_\phi \quad (20)$$

for every $\tau \geq 0$, where Ψ_τ is given by (18) and \mathcal{V}_ϕ is given by (17). Furthermore,

a) \mathcal{V}_ϕ is a C^1 manifold with $T_{(s,0)}\mathcal{V}_\phi = \mathbb{R} \times E(s)$ for $s \geq 0$;

b) there is $K > 0$ such that for every $(s, \xi), (s, \bar{\xi}) \in G$ and $t \geq s$ we have

$$\|\Psi_{t-s}(p_{s,\xi}) - \Psi_{t-s}(p_{s,\bar{\xi}})\| \leq K \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^\varepsilon \|\xi - \bar{\xi}\| \quad (21)$$

$$\|\partial(\Psi_{t-s}(p_{s,\xi})) - \partial(\Psi_{t-s}(p_{s,\bar{\xi}}))\| \leq K \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^{2\varepsilon} \|\xi - \bar{\xi}\| \quad (22)$$

where $p_{s,\xi} = (s, \xi, \phi(s, \xi))$.

Since problem (5) is equivalent to the problem

$$x(t) = U(t, s)\xi + \int_s^t U(t, r)f(r, x(r), y(r)) dr, \quad (23)$$

$$y(t) = V(t, s)\eta + \int_s^t V(t, r)f(r, x(r), y(r)) dr, \quad (24)$$

to prove the invariance in (20) we should have

$$x(t, \xi) = U(t, s)\xi + \int_s^t U(t, r)f(r, x(r, \xi), \phi(r, x(r, \xi))) dr, \quad (25)$$

$$\phi(t, x(t, \xi)) = V(t, s)\phi(s, \xi) + \int_s^t V(t, r)f(r, x(r, \xi), \phi(r, x(r, \xi))) dr \quad (26)$$

for every $s \geq 0$, every $t \geq s$ and every $\xi \in E(s)$.

The proof of the theorem goes as follows: in Lemma 1 we prove, using Banach fixed point theorem in a suitable space \mathcal{B}_s of functions, that for every $\phi \in \mathcal{X}$, there is a unique function $x_\phi \in \mathcal{B}_s$ verifying (25); in Lemma 2 we estimate the distance between two solutions x_ϕ and x_ψ given by Lemma 1; then we establish in Lemma 3 the equivalence between (26) with $x = x_\phi$ and a different equation; after that, another application of the Banach fixed point theorem (this time in space \mathcal{X}) gives us a unique solution of the new equation and the theorem follows easily.

For $s \geq 0$ and $C > D$, we denote by $\mathcal{B} = \mathcal{B}_s$ the space of C^1 functions

$$x: [s, +\infty[\times E(s) \rightarrow X$$

that, for every $t \geq s$ and $\xi, \bar{\xi} \in E(s)$, verify the following conditions

$$x(t, \xi) \in E(t) \quad (27)$$

$$x(s, \xi) = \xi, \quad x(t, 0) = 0 \quad (28)$$

$$\|\partial x(t, \xi)\| \leq C \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^\varepsilon \quad (29)$$

$$\|\partial x(t, \xi) - \partial x(t, \bar{\xi})\| \leq C \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^{2\varepsilon} \|\xi - \bar{\xi}\| \quad (30)$$

The mean value theorem and (29) imply that

$$\|x(t, \xi) - x(t, \bar{\xi})\| \leq C \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^\varepsilon \|\xi - \bar{\xi}\| \quad (31)$$

for every $t \geq s$ and $\xi, \bar{\xi} \in E(s)$ and when $\bar{\xi} = 0$ we have the following estimate

$$\|x(t, \xi)\| \leq C \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^\varepsilon \|\xi\| \quad (32)$$

for every $t \geq s$ and $\xi \in E(s)$. In space \mathcal{B}_s we consider the metric induced by

$$\|x\|' = \sup \left\{ \frac{\mu(s)^a}{\mu(s)^\varepsilon \mu(t)^a} \frac{\|x(t, \xi)\|}{\|\xi\|} : t \geq s, \xi \in E(s) \setminus \{0\} \right\}. \quad (33)$$

Proposition 1. *The space \mathcal{B}_s is a complete metric space with the metric induced by (33).*

Proof. For $x \in \mathcal{B}_s$, $t \geq s$ and $r > 0$, we define a function $x^{t,r} : B_s(r) \rightarrow F(t)$ by

$$x^{t,r}(\xi) = x(t, \xi),$$

where $B_s(r)$ is the open ball of $E(s)$ centered at 0 and with radius r . Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in \mathcal{B}_s with respect to the metric induced by (33). Then the sequence $(x_n^{t,r})_{n \in \mathbb{N}}$ is a Cauchy sequence with respect the supremum norm in the space of bounded functions from $B_s(r)$ into $F(t)$. Hence, there is a function $x^{t,r} : B_s(r) \rightarrow F(t)$ such that $(x_n^{t,r})_{n \in \mathbb{N}}$ converges to $x^{t,r}$ in the space of bounded functions from $B_s(r)$ into $F(t)$ equipped with the supremum norm.

For each $\xi, \bar{\xi} \in B_s(r)$, by (32), (29) and (30), and denoting the first derivative by D , we obtain

$$\begin{aligned} \|x_n^{t,r}(\xi)\| &\leq C \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^\varepsilon r \\ \| (Dx_n^{t,r})(\xi) \| &\leq C \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^\varepsilon, \\ \| (Dx_n^{t,r})(\xi) - (Dx_n^{t,r})(\bar{\xi}) \| &\leq C \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^{2\varepsilon} \|\xi - \bar{\xi}\|. \end{aligned}$$

Denote by $C_b^{1,1}(B_s(r), F(t))$ the space of C^1 functions u , defined from $B_s(r)$ into $F(t)$, having Lipschitz derivative and such that $\|u\|_{1,1} \leq b$, where $\|\cdot\|_{1,1}$ is defined by

$$\|u\|_{1,1} = \max\{\|u\|_\infty, \|Du\|_\infty, L(Du)\},$$

$\|\cdot\|_\infty$ is the supremum norm and

$$L(u) = \sup \left\{ \frac{\|u(\xi) - u(\bar{\xi})\|}{\|\xi - \bar{\xi}\|} : \xi, \bar{\xi} \in B_s(r) \text{ with } \xi \neq \bar{\xi} \right\}.$$

With

$$b = C \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^\varepsilon \max\{\mu(s)^\varepsilon, r\},$$

it follows that $x_n^{t,r} \in C_b^{1,1}(B_s(r), F(t))$.

From the generalization of Henry's Lemma (see [15, p.151]) given by Elbialy [14] (for related results see also [16] and [13]) we conclude that $x^{t,r} \in C_b^{1,1}(B_s(r), F(t))$ and

$$(Dx_n^{t,r})_{n \in \mathbb{N}} \text{ converges pointwise to } Dx^{t,r} \text{ when } n \rightarrow \infty \quad (34)$$

for every $\xi \in B_s(r)$. The uniqueness of each function $x^{t,r}$ in the ball $B_s(r)$ implies that we can obtain a function $x : [s, +\infty[\times E(s) \rightarrow X$ such that $x(t, \xi) = x^{t,r}(\xi)$ for each $r > 0$, $t \geq s$ and $\xi \in B_s(r)$. From (34) we can easily see that $x \in \mathcal{B}_s$. Furthermore, because $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, for each $\kappa > 0$ there is $p \in \mathbb{N}$ such that for $n, m > p$ we have

$$\|x_n(t, \xi) - x_m(t, \xi)\| \leq \kappa \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^\varepsilon \|\xi\| \quad (35)$$

for every $t \geq s$ and $\xi \in E(s)$. Letting $m \rightarrow \infty$ in (35) we get

$$\|x_n(t, \xi) - x(t, \xi)\| \leq \kappa \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^\varepsilon \|\xi\|,$$

and therefore $(x_n)_{n \in \mathbb{N}}$ converges to x in the space \mathcal{B}_s . \square

We equip the space \mathcal{X} with the metric induced by

$$\|\phi\|' = \sup \left\{ \frac{\|\phi(s, \xi)\|}{\|\xi\|} : (s, \xi) \in G \right\}. \quad (36)$$

Proposition 2. *The space \mathcal{X} is a complete metric space with the metric induced by (36).*

The proof of Proposition 2 is similar to the proof of Proposition 1 and therefore is omitted.

Define

$$\phi_x(r, \xi) = \phi(r, x(r, \xi)) \quad \text{and} \quad f_{x, \phi}(r, \xi) = f(t, x(r, \xi), \phi_x(r, \xi))$$

for each $x \in \mathcal{B}_s$ and $\phi \in \mathcal{X}$.

Lemma 1. *Let $s \geq 0$ and $\phi \in \mathcal{X}$. For $\delta > 0$ sufficiently small, there is one and only one $x = x_\phi \in \mathcal{B}_s$ such that*

$$x(t, \xi) = U(t, s)\xi + \int_s^t U(t, r) f_{x, \phi}(r, \xi) dr \quad (37)$$

for every $t \geq s$ and $\xi \in E(s)$.

Proof. Given $\phi \in \mathcal{X}$, consider in \mathcal{B}_s the operator $J = J_\phi$ given, for every $x \in \mathcal{B}_s$, by

$$(Jx)(t, \xi) = U(t, s)\xi + \int_s^t U(t, r) f_{x, \phi}(r, \xi) dr$$

for each $(t, \xi) \in [s, +\infty[\times E(s)$. First we will prove that $Jx \in \mathcal{B}_s$ for every $x \in \mathcal{B}_s$.

The definition of J immediately assures that $(Jx)(s, \xi) = U(s, s)\xi = \xi$ for every $\xi \in E(s)$ and that $(Jx)(t, \xi) \in E(t)$ for every $t \geq s$ and every $\xi \in E(s)$. Furthermore, from (28), (13) and (6) we obtain $(Jx)(t, 0) = 0$ for every $t \geq s$.

Moreover, the operator J is of class C^1 and

$$\partial(Jx)(t, \xi) = U(t, s) + \int_s^t U(t, r) \partial f_{x, \phi}(r, \xi) dr$$

and this implies that

$$\|\partial(Jx)(t, \xi)\| \leq \|U(t, s)\| + \int_s^t \|U(t, r)\| \|\partial f_{x, \phi}(r, \xi)\| dr. \quad (38)$$

From the chain rule and (14) it follows that

$$\begin{aligned} \|\partial f_{x, \phi}(r, \xi)\| &\leq \|\partial f(r, x(r, \xi), \phi(r, x(r, \xi)))\| [\|\partial x(r, \xi)\| + \|\partial \phi(r, x(r, \xi))\| \|\partial x(r, \xi)\|] \\ &\leq 2\|\partial f(r, x(r, \xi), \phi(r, x(r, \xi)))\| \|\partial x(r, \xi)\| \end{aligned}$$

for every $r \geq s$ and every $\xi \in E(s)$. By (7) and (29) we obtain

$$\|\partial f_{x, \phi}(r, \xi)\| \leq 2C\delta\mu'(r)\mu(r)^{-3\varepsilon-1} \left[\frac{\mu(r)}{\mu(s)} \right]^a \mu(s)^\varepsilon. \quad (39)$$

for every $r \geq s$ and every $\xi \in E(s)$. Using (2) and (39) we get

$$\begin{aligned} \int_s^t \|U(t, r)\| \|\partial f_{x, \phi}(r, \xi)\| dr &\leq 2CD\delta \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^\varepsilon \int_s^t \mu'(r) \mu(r)^{-2\varepsilon-1} dr \\ &\leq \frac{CD\delta}{\varepsilon} \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^{-\varepsilon}. \end{aligned}$$

Using this inequality, (2) and (38) we have

$$\|\partial(Jx)(t, \xi)\| \leq \left(D + \frac{CD\delta}{\varepsilon} \right) \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^\varepsilon$$

for every $t \geq s$ and every $\xi \in E(s)$. Choosing $\delta \leq \varepsilon \left(\frac{1}{D} - \frac{1}{C} \right)$ we obtain for every $t \geq s$ and every $\xi \in E(s)$

$$\|\partial(Jx)(t, \xi)\| \leq C \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^\varepsilon.$$

For $r \geq s$ and $\xi \in E(s)$, using again the chain rule, (7), (8), (14), (15) and (16), we have

$$\begin{aligned} &\|\partial f_{x, \phi}(r, \xi) - \partial f_{x, \phi}(r, \bar{\xi})\| \\ &\leq \|\partial f(r, x(r, \xi), \phi_x(r, \xi)) - \partial f(r, x(r, \bar{\xi}), \phi_x(r, \bar{\xi}))\| \times \\ &\quad \times (\|\partial x(r, \xi)\| + \|\partial \phi(r, x(r, \xi))\| \|\partial x(r, \xi)\|) \\ &\quad + \|\partial f(r, x(r, \bar{\xi}), \phi_x(r, \bar{\xi}))\| \|\partial x(r, \xi) - \partial x(r, \bar{\xi})\| \\ &\quad + \|\partial f(r, x(r, \bar{\xi}), \phi_x(r, \bar{\xi}))\| \|\partial \phi(r, x(r, \xi))\| \|\partial x(r, \xi) - \partial x(r, \bar{\xi})\| \\ &\quad + \|\partial f(r, x(r, \bar{\xi}), \phi_x(r, \bar{\xi}))\| \|\partial \phi(r, x(r, \bar{\xi})) - \partial \phi(r, x(r, \xi))\| \|\partial x(r, \bar{\xi})\| \\ &\leq 2\delta\mu'(r) \mu(r)^{-3\varepsilon-1} (\|x(r, \xi) - x(r, \bar{\xi})\| + \|\phi_x(r, \xi) - \phi_x(r, \bar{\xi})\|) \|\partial x(r, \xi)\| \\ &\quad + 2\delta\mu'(r) \mu(r)^{-3\varepsilon-1} \|\partial x(r, \xi) - \partial x(r, \bar{\xi})\| \\ &\quad + \delta\mu'(r) \mu(r)^{-3\varepsilon-1} \|x(r, \xi) - x(r, \bar{\xi})\| \|\partial x(r, \bar{\xi})\| \\ &\leq \delta\mu'(r) \mu(r)^{-3\varepsilon-1} (2\|\partial x(r, \xi) - \partial x(r, \bar{\xi})\| + 5\|x(r, \xi) - x(r, \bar{\xi})\| \|\partial x(r, \bar{\xi})\|) \end{aligned}$$

and by (30), (31) and (29) we obtain

$$\|\partial f_{x, \phi}(r, \xi) - \partial f_{x, \phi}(r, \bar{\xi})\| \leq 7C^2\delta\mu'(r) \mu(r)^{-3\varepsilon-1} \left[\frac{\mu(r)}{\mu(s)} \right]^a \mu(s)^{2\varepsilon} \|\xi - \bar{\xi}\|. \quad (40)$$

Therefore, for every $t \geq s$ and every $\xi \in E(s)$, we obtain

$$\begin{aligned} \|\partial(Jx)(t, \xi) - \partial(Jx)(t, \bar{\xi})\| &\leq \int_s^t \|U(t, r)\| \|\partial f_{x, \phi}(r, \xi) - \partial f_{x, \phi}(r, \bar{\xi})\| dr \\ &\leq 7C^2D\delta \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^{2\varepsilon} \|\xi - \bar{\xi}\| \int_s^t \mu'(r) \mu(r)^{-2\varepsilon-1} dr \\ &\leq \frac{7C^2D\delta}{2\varepsilon} \left[\frac{\mu(t)}{\mu(s)} \right]^a \|\xi - \bar{\xi}\| \end{aligned}$$

and for $\delta < \frac{2\varepsilon}{7CD}$ we have

$$\|\partial(Jx)(t, \xi) - \partial(Jx)(t, \bar{\xi})\| \leq C \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^{2\varepsilon} \|\xi - \bar{\xi}\|.$$

We have proved Jx verifies (27), (28), (29) and (30) for every $x \in \mathcal{B}_s$. Therefore J is an operator from \mathcal{B}_s into \mathcal{B}_s .

Now we will prove that, choosing δ sufficiently small, J is a contraction in \mathcal{B}_s . From (9) and (16) we have for $r \geq s$ and $\xi \in E(s)$

$$\begin{aligned} \|f_{x,\phi}(r, \xi) - f_{y,\phi}(r, \xi)\| &\leq \delta \mu'(r) \mu(r)^{-3\varepsilon-1} \|(x(r, \xi), \phi_x(r, \xi)) - (y(r, \xi), \phi_y(r, \xi))\| \\ &\leq 2\delta \mu'(r) \mu(r)^{-3\varepsilon-1} \|x(r, \xi) - y(r, \xi)\| \\ &\leq 2\delta \mu'(r) \mu(r)^{-3\varepsilon-1} \left[\frac{\mu(r)}{\mu(s)} \right]^a \mu(s)^\varepsilon \|\xi\| \|x - y\|. \end{aligned}$$

Using this estimate, we obtain by (2), for every $t \geq s$ and every $\xi \in E(s)$,

$$\begin{aligned} \|(Jx)(t, \xi) - (Jy)(t, \xi)\| &\leq \int_s^t \|U(t, r)\| \|f_{x,\phi}(r, \xi) - f_{y,\phi}(r, \xi)\| dr \\ &\leq 2D\delta \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^\varepsilon \|\xi\| \|x - y\| \int_s^t \mu'(r) \mu(r)^{-2\varepsilon-1} dr \\ &\leq \frac{D\delta}{\varepsilon} \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^\varepsilon \|\xi\| \|x - y\| \end{aligned}$$

and thus

$$\|Jx - Jy\|' \leq \frac{D\delta}{\varepsilon} \|x - y\|'$$

for every $x, y \in \mathcal{B}_s$. Therefore, choosing $\delta < \varepsilon/D$, we conclude that J is a contraction. Because \mathcal{X} is a complete metric space, J has a unique fixed point $x_\phi \in \mathcal{B}_s$ and this fixed point verifies (37). This concludes the proof. \square

Given $\phi \in \mathcal{X}$ we denote by x_ϕ the unique function in \mathcal{B}_s that verifies (37). In the next Lemma we obtain an estimate that, in the exponential case, is usually obtained using Gronwall's lemma. Here we use an induction argument that allows us to obtain a corresponding estimate in our generalized context.

Lemma 2. *Choosing $\delta > 0$ sufficiently small, we have*

$$\|x_\phi(t, \xi) - x_\psi(t, \xi)\| \leq C \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^{-\varepsilon} \|\xi\| \cdot \|\phi - \psi\|' \quad (41)$$

for every $\phi, \psi \in \mathcal{X}$ and every $(t, \xi) \in [s, +\infty[\times E(s)$.

Proof. Given $\phi, \psi \in \mathcal{X}$, we write $y_{n+1} = J_\phi y_n$ and $z_{n+1} = J_\psi z_n$ with

$$y_1(t, \xi) = z_1(t, \xi) = U(t, s)\xi$$

for every $t \geq s$ and every $\xi \in E(s)$. Since x_ϕ and x_ψ were obtained in Lemma 1 using Banach's fixed point theorem, it follows that

$$\|x_\phi - x_\psi\|' = \lim_{n \rightarrow +\infty} \|y_n - z_n\|'.$$

Hence, to prove (41), it is enough to prove that, for each $n \in \mathbb{N}$, we have

$$\|y_n(t, \xi) - z_n(t, \xi)\| \leq C \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^{-\varepsilon} \|\xi\| \cdot \|\phi - \psi\|' \quad (42)$$

for all $(t, \xi) \in [s, +\infty[\times E(s)$. We are going to prove inequality (42) by mathematical induction on n . Obviously, inequality (42) holds for $n = 1$. Suppose that (42)

holds for n . Then, for $r \geq s$ and $\xi \in E(s)$, we have by (8), (32) and the induction hypothesis

$$\begin{aligned}
& \|f_{y_n, \phi}(r, \xi) - f_{z_n, \psi}(r, \xi)\| \\
& \leq \delta \mu'(r) \mu(r)^{-3\varepsilon-1} (\|y_n(r, \xi) - z_n(r, \xi)\| + \|\phi_{y_n}(r, \xi) - \psi_{z_n}(r, \xi)\|) \\
& \leq \delta \mu'(r) \mu(r)^{-3\varepsilon-1} (2\|y_n(r, \xi) - z_n(r, \xi)\| + \|\phi_{z_n}(r, \xi) - \psi_{z_n}(r, \xi)\|) \\
& \leq \delta \mu'(r) \mu(r)^{-3\varepsilon-1} (2\|y_n(r, \xi) - z_n(r, \xi)\| + \|\phi - \psi\|' \|z_n(r, \xi)\|) \\
& \leq 3\delta C \mu'(r) \mu(r)^{-3\varepsilon-1} \left[\frac{\mu(r)}{\mu(s)} \right]^a \mu(s)^\varepsilon \|\xi\| \cdot \|\phi - \psi\|'
\end{aligned}$$

and this implies that

$$\begin{aligned}
& \|y_{n+1}(t, \xi) - z_{n+1}(t, \xi)\| \\
& \leq \int_s^t \|U(t, r)\| \|f_{y_n, \phi}(r, \xi) - f_{z_n, \psi}(r, \xi)\| dr \\
& \leq 3CD\delta \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^\varepsilon \|\xi\| \cdot \|\phi - \psi\|' \int_s^t \mu'(r) \mu(r)^{-2\varepsilon-1} dr \\
& \leq \frac{3CD\delta}{2\varepsilon} \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^{-\varepsilon} \|\xi\| \cdot \|\phi - \psi\|'
\end{aligned}$$

for every $t \geq s$ and every $\xi \in E(s)$. Thus, choosing $\delta < \frac{2\varepsilon}{3D}$, inequality (42) holds for $n+1$. Therefore (42) is true for all $n \in \mathbb{N}$ and this finishes the proof of the lemma. \square

Lemma 3. *If $\delta > 0$ is sufficiently small, for every $\phi \in \mathcal{X}$, the following properties are equivalent*

a) *for every $s \geq 0$, $t \geq s$ and $\xi \in E(s)$,*

$$\phi_{x_\phi}(t, \xi) = V(t, s)\phi(s, \xi) + \int_s^t V(t, r)f_{x_\phi, \phi}(r, \xi) dr, \quad (43)$$

where $x_\phi \in \mathcal{B}_s$ is given by Lemma 1;

b) *for every $s \geq 0$ and every $\xi \in E(s)$,*

$$\phi(s, \xi) = - \int_s^{+\infty} V(r, s)^{-1} f_{x_\phi, \phi}(r, \xi) dr, \quad (44)$$

where $x_\phi \in \mathcal{B}_s$ is given by Lemma 1.

Proof. We start by showing that the integral in (44) is well defined. For every $r \geq s$, from (10) and using the fact that $x_\phi \in \mathcal{B}_s$, we obtain

$$\begin{aligned}
\|f_{x_\phi, \phi}(r, \xi)\| & \leq 2\delta \mu'(r) \mu(r)^{-3\varepsilon-1} \|x_\phi(r, \xi)\| \\
& \leq 2C\delta \mu'(r) \mu(r)^{-3\varepsilon-1} \left[\frac{\mu(r)}{\mu(s)} \right]^a \mu(s)^\varepsilon \|\xi\|
\end{aligned}$$

and by (3) this implies that

$$\begin{aligned} & \int_s^{+\infty} \|V(r, s)^{-1}\| \|f_{x_\phi, \phi}(r, \xi)\| dr \\ & \leq 2CD\delta\mu(s)^{-a+b+\varepsilon}\|\xi\| \int_s^{+\infty} \mu'(r)\mu(r)^{a-b-2\varepsilon-1} dr \\ & \leq \frac{2CD\delta}{|a-b-2\varepsilon|}\mu(s)^{-\varepsilon}\|\xi\|. \end{aligned}$$

Therefore the integral in (44) is well defined.

Now let us assume that a) is verified. Then, from (43) we get

$$\phi(s, \xi) = V(t, s)^{-1}\phi_{x_\phi}(t, \xi) - \int_s^t V(r, s)^{-1}f_{x_\phi, \phi}(r, \xi) dr. \quad (45)$$

Since

$$\begin{aligned} \|V(t, s)^{-1}\phi_{x_\phi}(t, \xi)\| & \leq D \left[\frac{\mu(t)}{\mu(s)} \right]^{-b} \mu(t)^\varepsilon \|x_\phi(t, \xi)\| \\ & \leq D \left[\frac{\mu(t)}{\mu(s)} \right]^{-b} \mu(t)^\varepsilon C \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^\varepsilon \|\xi\| \\ & = DC\|\xi\|\mu(s)^{-a+b+\varepsilon}\mu(t)^{a-b+\varepsilon}, \end{aligned}$$

we can conclude from (19) that $V(t, s)^{-1}\phi_{x_\phi}(t, \xi)$ converges to zero as t converges to $+\infty$. Therefore we obtain b) letting $t \rightarrow +\infty$ in (45).

Assume now that b) holds. Defining a semiflow F_r on G by

$$F_r(s, \xi) = (s + r, x_\phi(s + r, \xi)),$$

we obtain from (44)

$$\phi(s, \xi) = - \int_s^{+\infty} V(r, s)^{-1}f(F_{r-s}(s, \xi), \phi(F_{r-s}(s, \xi))) dr. \quad (46)$$

Replacing (s, ξ) by $(t, x_\phi(t, \xi))$ in (46) we get

$$\begin{aligned} \phi(t, x_\phi(t, \xi)) & = - \int_t^{+\infty} V(r, t)^{-1}f(F_{r-t}(t, x_\phi(t, \xi)), \phi(F_{r-t}(t, x_\phi(t, \xi)))) dr \\ & = - \int_t^{+\infty} V(r, t)^{-1}f(r, x_\phi(r, \xi), \phi(r, x_\phi(t, \xi))) dr, \end{aligned}$$

because F_r is a semiflow and this implies

$$F_{r-t}(t, x_\phi(t, \xi)) = F_{r-t}(F_{t-s}(s, \xi)) = F_{r-s}(s, \xi) = (r, x_\phi(r, \xi)).$$

Then, since $V(t, s)V(r, s)^{-1} = V(t, r)$, we obtain

$$\begin{aligned} V(t, s)\phi(s, \xi) & = - \int_s^{+\infty} V(t, s)V(r, s)^{-1}f_{x_\phi, \phi}(r, \xi) dr \\ & = - \int_s^t V(t, r)f_{x_\phi, \phi}(r, \xi) dr - \int_t^{+\infty} V(t, r)f_{x_\phi, \phi}(r, \xi) dr \\ & = - \int_s^t V(t, r)f_{x_\phi, \phi}(r, \xi) dr + \phi(t, x_\phi(t, \xi)) \end{aligned}$$

and thus

$$V(t, s)\phi(s, \xi) + \int_s^t V(t, r)f_{x_\phi, \phi}(r, \xi) dr = \phi(t, x_\phi(t, \xi)).$$

Hence $b)$ implies $a)$. \square

Next we will prove, for δ sufficiently small, the existence of a unique function $\phi \in \mathcal{X}$ that verifies (44).

Lemma 4. *Choosing $\delta > 0$ sufficiently small, there is a unique $\phi \in \mathcal{X}$ such that (44) holds for every $s \geq 0$ and every $\xi \in E(s)$.*

Proof. Let Φ be an operator on \mathcal{X} defined by

$$(\Phi\phi)(s, \xi) = - \int_s^{+\infty} V(r, s)^{-1} f_{x_\phi, \phi}(r, \xi) dr$$

for each $\phi \in \mathcal{X}$ and each $(s, \xi) \in G$. First we will prove that Φ is an operator from \mathcal{X} into \mathcal{X} .

Obviously, $(\Phi\phi)(s, \xi) \in F(s)$ for every $(s, \xi) \in G$. Furthermore, $(\Phi\phi)(s, 0) = 0$ for every $s \geq 0$ because $x_\phi(r, 0) = 0$ for every $\phi \in \mathcal{X}$ and every $r \geq s$. Moreover, $\Phi\phi$ is of class C^1 and

$$\partial(\Phi\phi)(s, \xi) = - \int_s^{+\infty} V(r, s)^{-1} \partial f_{x_\phi, \phi}(r, \xi) dr.$$

By (6) we have $\partial(\Phi\phi)(s, 0) = 0$ for every $s \geq 0$.

From (3) and (39) we have

$$\begin{aligned} \|\partial(\Phi\phi)(s, \xi)\| &\leq \int_s^{+\infty} \|V(r, s)^{-1}\| \cdot \|\partial f_{x_\phi, \phi}(r, \xi)\| dr \\ &\leq 2CD\delta\mu(s)^{-a+b+\varepsilon} \int_s^{+\infty} \mu'(r)\mu(r)^{a-b-2\varepsilon-1} dr \\ &\leq \frac{2CD\delta}{|a-b-2\varepsilon|} \end{aligned}$$

and choosing $\delta < \frac{|a-b-2\varepsilon|}{2CD}$ we obtain for every $s \geq 0$ and every $\xi \in E(s)$

$$\|\partial(\Phi\phi)(s, \xi)\| \leq 1.$$

It follows from (3) and (40) that

$$\begin{aligned} &\|\partial(\Phi\phi)(s, \xi) - \partial(\Phi\phi)(s, \bar{\xi})\| \\ &\leq \int_s^{+\infty} \|V(r, s)^{-1}\| \cdot \|\partial f_{x_\phi, \phi}(r, \xi) - \partial f_{x_\phi, \phi}(r, \bar{\xi})\| dr \\ \mu(s)^{2\varepsilon} &\leq 7C^2 D\delta\mu(s)^{-a+b+2\varepsilon} \|\xi - \bar{\xi}\| \int_s^{+\infty} \mu'(r)\mu(r)^{a-b-2\varepsilon-1} dr \\ &= \frac{7C^2 D\delta}{|a-b-2\varepsilon|} \|\xi - \bar{\xi}\|. \end{aligned}$$

Therefore, if $\delta \leq \frac{|a-b-2\varepsilon|}{7C^2 D}$, we get

$$\|\partial(\Phi\phi)(s, \xi) - \partial(\Phi\phi)(s, \bar{\xi})\| \leq \|\xi - \bar{\xi}\|$$

for every $s \geq 0$ and every $\xi, \bar{\xi} \in E(s)$.

Hence, $\Phi\phi$ satisfies (12), (13), (14) and (15) for every $\phi \in \mathcal{X}$ and this proves that Φ is an operator from \mathcal{X} into \mathcal{X} .

To finish the proof we will verify that Φ is a contraction for δ sufficiently small. Let $\phi, \psi \in \mathcal{X}$ and $(s, \xi) \in \mathbb{R}_0^+ \times E(s)$. By (9), (16), (41) and (32), for $r \geq s$, we have

$$\begin{aligned} & \|f_{x_\phi, \phi}(r, \xi) - f_{x_\psi, \psi}(r, \xi)\| \\ & \leq \delta \mu'(r) \mu(r)^{-3\varepsilon-1} (\|x_\phi(r, \xi) - x_\psi(r, \xi)\| + \|\phi_{x_\phi}(r, \xi) - \psi_{x_\psi}(r, \xi)\|) \\ & \leq \delta \mu'(r) \mu(r)^{-3\varepsilon-1} (2\|x_\phi(r, \xi) - x_\psi(r, \xi)\| + \|\phi_{x_\phi}(r, \xi) - \psi_{x_\phi}(r, \xi)\|) \\ & \leq \delta \mu'(r) \mu(r)^{-3\varepsilon-1} (2\|x_\phi(r, \xi) - x_\psi(r, \xi)\| + \|\phi - \psi\|' \|x_\phi(r, \xi)\|) \\ & \leq 3CD\delta \mu'(r) \mu(r)^{-3\varepsilon-1} \left[\frac{\mu(r)}{\mu(s)} \right]^a \mu(s)^\varepsilon \|\xi\| \cdot \|\phi - \psi\|'. \end{aligned}$$

The last inequality and (3) implies that

$$\begin{aligned} & \|(\Phi\phi)(s, \xi) - \Phi\psi(s, \xi)\| \\ & \leq \int_s^{+\infty} \|V(r, s)^{-1}\| \cdot \|f_{x_\phi, \phi}(r, \xi) - f_{x_\psi, \psi}(r, \xi)\| dr \\ & \leq 3CD\delta \|\xi\| \cdot \|\phi - \psi\|' \mu(s)^{-a+b+\varepsilon} \int_s^{+\infty} \mu'(r) \mu(r)^{a-b-2\varepsilon-1} dr \\ & \leq \frac{3CD\delta}{|a-b-2\varepsilon|} \|\xi\| \cdot \|\phi - \psi\|' \end{aligned}$$

and this implies that

$$\|\Phi\phi - \Phi\psi\|' \leq \frac{3CD\delta}{|a-b-2\varepsilon|} \|\phi - \psi\|'.$$

Therefore, if $\delta < \frac{|a-b-2\varepsilon|}{3CD}$, we conclude that Φ is a contraction in \mathcal{X} and that the unique fixed point of Φ verifies (44). \square

Proof of Theorem 1. By Lemma 1, for each $\phi \in \mathcal{X}$ there is a unique sequence $x_\phi \in \mathcal{B}_s$ satisfying identity (25). By Lemma (3), solving equation (26) with $x = x_\phi$ is equivalent to solve equation (44). Finally, by Lemma 4, there is a unique solution of (44). Therefore, we obtain a unique solution of equation (26) with $x = x_\phi$ for δ sufficiently small.

To prove that, for the function ϕ that solves (26) with $x = x_\phi$, the graph \mathcal{V}_ϕ is a C^1 manifold we have to consider the map

$$S: \mathbb{R}_0^+ \times E(0) \rightarrow \mathbb{R}_0^+ \times X$$

defined by

$$S(t, \xi) = \Psi_t(0, \xi, \phi(0, \xi)).$$

The map S is of class C^1 because $\phi(0, \xi)$ is also of class C^1 . Moreover, if $S(t, \xi) = S(t', \xi')$, then $t = t'$ and $\xi = \xi'$. Thus, S is a parametrization of class C^1 of the set \mathcal{V}_ϕ . Therefore, \mathcal{V}_ϕ is a C^1 manifold.

Finally, for every $(s, \xi), (s, \bar{\xi}) \in G$ and every $t \geq s$, from (16) we have

$$\begin{aligned} \|\Psi_{t-s}(p_{s,\xi}) - \Psi_{t-s}(p_{s,\bar{\xi}})\| &= \|(t, x_\phi(t, \xi), \phi_{x_\phi}(t, \xi)) - (t, x_\phi(t, \bar{\xi}), \phi_{x_\phi}(t, \bar{\xi}))\| \\ &\leq \|x_\phi(t, \xi) - x_\phi(t, \bar{\xi})\| + \|\phi_{x_\phi}(t, \xi) - \phi_{x_\phi}(t, \bar{\xi})\| \\ &\leq 2\|x_\phi(t, \xi) - x_\phi(t, \bar{\xi})\| \\ &\leq 2C \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^\varepsilon \|\xi - \bar{\xi}\| \end{aligned}$$

and this proves (21). To prove (22), we note that

$$\begin{aligned} \|\partial\phi_{x_\phi}(t, \xi) - \partial\phi_{x_\phi}(t, \bar{\xi})\| &\leq \|\partial\phi(t, x_\phi(t, \xi)) - \partial\phi(t, x_\phi(t, \bar{\xi}))\| \cdot \|\partial x_\phi(t, \xi)\| \\ &\quad + \|\partial\phi(t, x_\phi(t, \bar{\xi}))\| \cdot \|\partial x_\phi(t, \xi) - \partial x_\phi(t, \bar{\xi})\| \\ &\leq \|x_\phi(t, \xi) - x_\phi(t, \bar{\xi})\| \cdot \|\partial x_\phi(t, \xi)\| \\ &\quad + \|\partial x_\phi(t, \xi) - \partial x_\phi(t, \bar{\xi})\| \\ &\leq (C + C^2) \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^{2\varepsilon} \|\xi - \bar{\xi}\|. \end{aligned}$$

This last estimate is a consequence of the chain rule, (15), (14), (31), (29) and (30). Therefore, by the last estimate and (30), we have

$$\begin{aligned} \|\partial\Psi_{t-s}(p_{s,\xi}) - \partial\Psi_{t-s}(p_{s,\bar{\xi}})\| &= \|\partial(t, x_\phi(t, \xi), \phi_{x_\phi}(t, \xi)) - \partial(t, x_\phi(t, \bar{\xi}), \phi_{x_\phi}(t, \bar{\xi}))\| \\ &\leq \|\partial x_\phi(t, \xi) - \partial x_\phi(t, \bar{\xi})\| + \|\partial\phi_{x_\phi}(t, \xi) - \partial\phi_{x_\phi}(t, \bar{\xi})\| \\ &\leq (2C + C^2) \left[\frac{\mu(t)}{\mu(s)} \right]^a \mu(s)^{2\varepsilon} \|\xi - \bar{\xi}\|. \end{aligned}$$

This completes the proof of the theorem. \square

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